[Inverse Trigonometric Functions]

(प्रतिलोम त्रिकोणमितीय फलन)

Inverse Function

$f : A \rightarrow B$

Bijective

One-one	onto

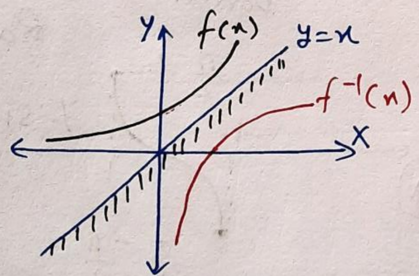
$f^{-1} : B \rightarrow A$

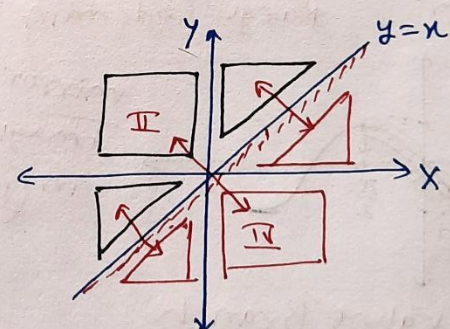
Bijective

one-one	onto

- Domain of $f = A$ = Range of f^{-1}
- Range of $f = B$ = Domain of f^{-1}

- $f^{-1} \circ f(x) = x \,,\; x \in A$
- $f \circ f^{-1}(x) = x \,,\; x \in B$
- Graphs of f and f^{-1} are mirror image of each other in the mirror line $y = x$.

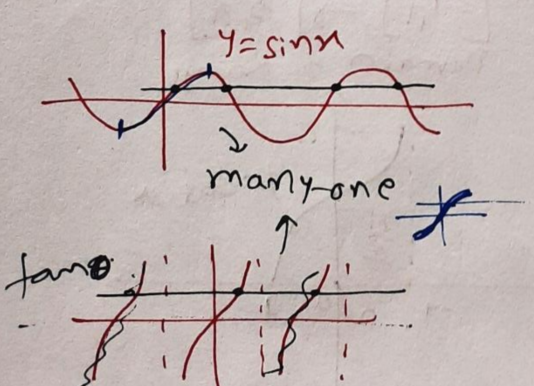




Trigonometric Functions :→ $f(x) = y = \sin x$

y ↑ Real Value (Output)

x ↑ Angle (input)



- Trigonometric Functions are not one-one in their Domain.
- But these functions can be made one-one - onto by modifying Domain & Co-Domain.

$f^{-1} \neq \frac{1}{f}$ $\sin^{-1}x \neq \frac{1}{\sin x}$

Inverse Trigonometric Functions: $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$.

$f(x) = y = \sin^{-1}x$

y ↓ Angle (Output)

x ↓ Real value (input)

$\boxed{\begin{array}{l} f(x)= y = \sin x \text{ (Trigo.)} \\ \sin^{-1}(y) = x \end{array}}$

$\sin^{-1}(x) = f^{-1}(x)$

e.g. $\frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

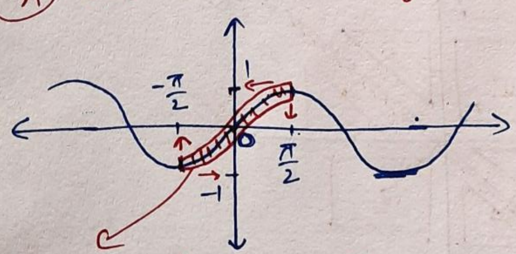
Graphs, Domain, Range, Principal Value Branch (PVB)

Principal Value Branch ↓ Bijective (one-one, onto)

1 $\boxed{y = f(x) = \sin x}$

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$

(x) Domain Range = Codomain



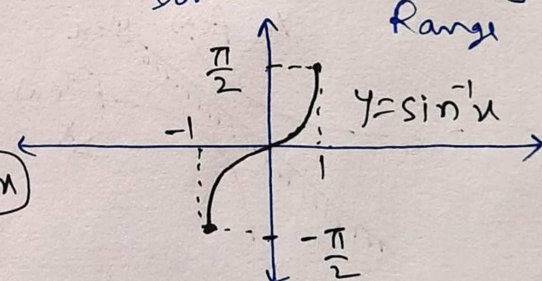
Principal value Branch

mirror image in $y = x$ →

$\boxed{y = f^{-1}(x) = \sin^{-1}x}$

$[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain Range



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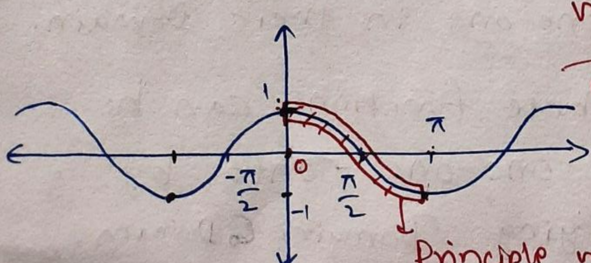
2 $\boxed{y = f(x) = \cos x}$

$[0, \pi] \rightarrow [-1, 1]$

(Bijective)



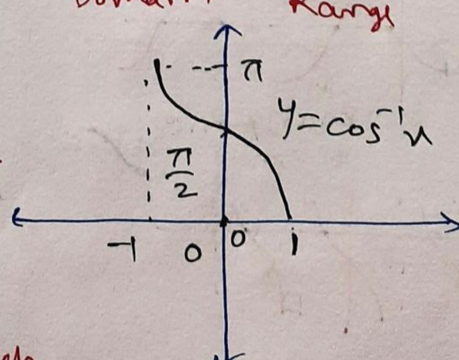
Principle value Branch

mirror image ($y = x$) →

$\boxed{y = f^{-1}(x) = \cos^{-1}x}$

$[-1, 1] \rightarrow [0, \pi]$

Domain Range

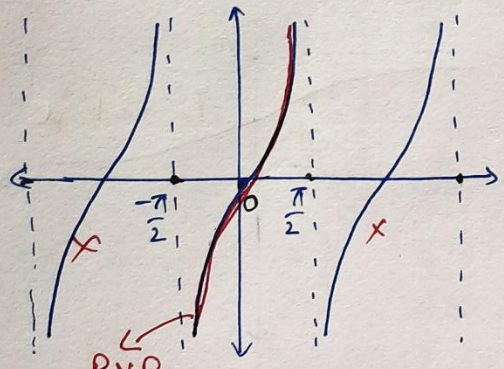


3 $\boxed{y = f(x) = \tan x}$

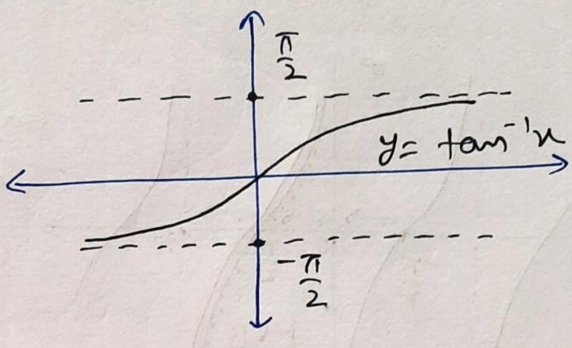
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$ R

$\boxed{y = f^{-1}(x) = \tan^{-1} x}$

$R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



mirror image



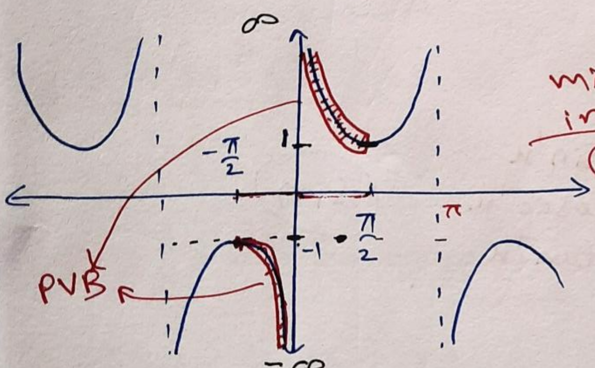
4 $\boxed{y = f(x) = \operatorname{cosec} x}$

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to (-\infty, -1] \cup [1, \infty)$

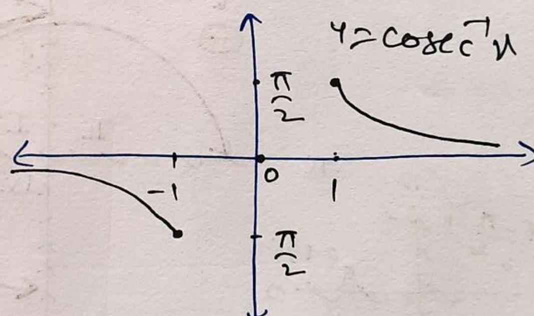
$\boxed{y = f^{-1}(x) = \operatorname{cosec}^{-1} x}$

$(-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Domain Range



mirror image ($y = x$)



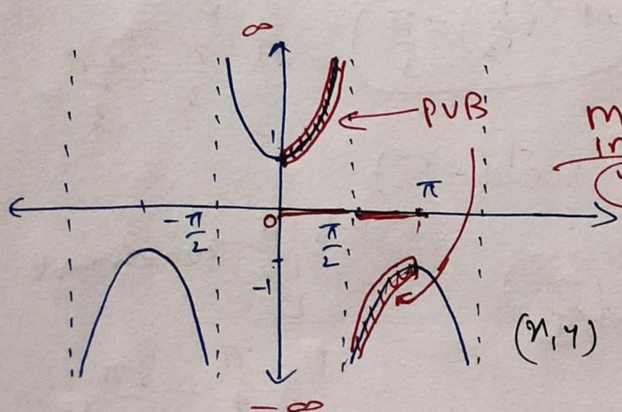
5 $y = f(x) = \sec x$

$[0, \pi] - \left\{\frac{\pi}{2}\right\} \to (-\infty, -1] \cup [1, \infty)$

$\boxed{y = f^{-1}(x) = \sec^{-1} x}$

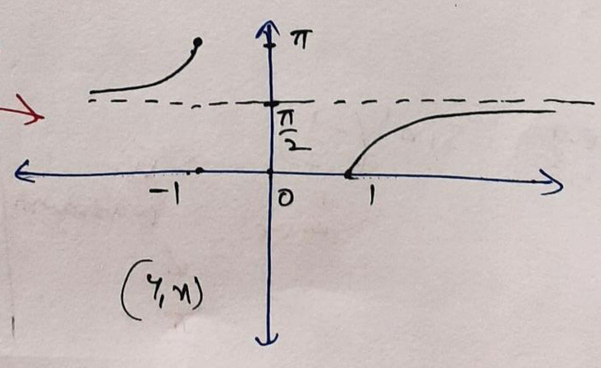
$(-\infty, -1] \cup [1, \infty) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Domain Range



(x, y)

mirror image ($y = x$)



(y, x)

6 $\boxed{y = \cot x = f(x)}$

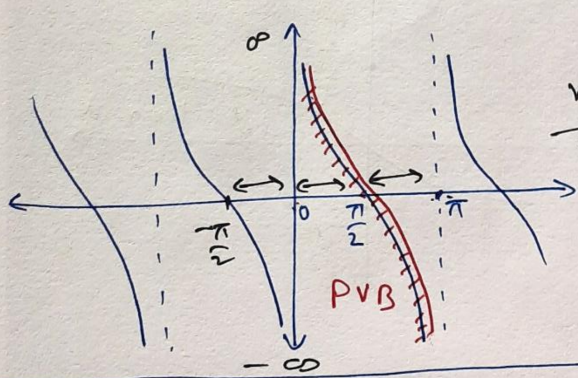
$(0, \pi) \to (-\infty, \infty)$ (R)

$\boxed{y = \cot^{-1} x = f^{-1}(x)}$

$(-\infty, \infty) \to (0, \pi)$

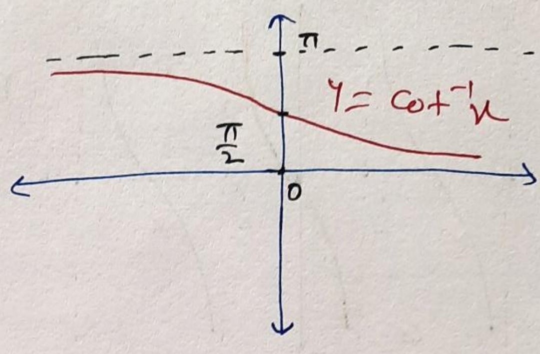
R

Domain

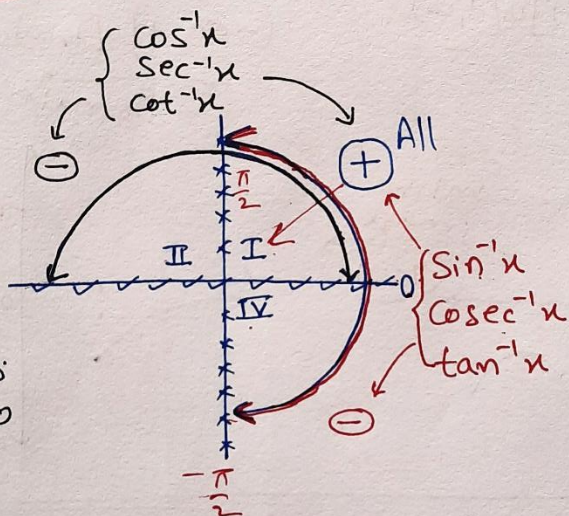


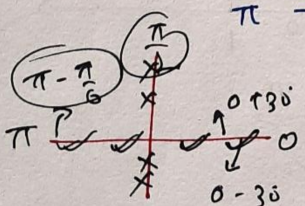
mirror image

$y = x$



How to Solve Questions Fast ??





sin

e.g. $\sin^{-1}\left(\frac{1}{2}\right) = x = \frac{\pi}{6}$

$\frac{1}{2} = \sin x$

I

e.g. $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) \to$ II-quadrant

$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ ✓

e.g. $\sin^{-1}\left(-\frac{1}{2}\right) \to$ IV quadrant

−ve

$-\frac{\pi}{6}$

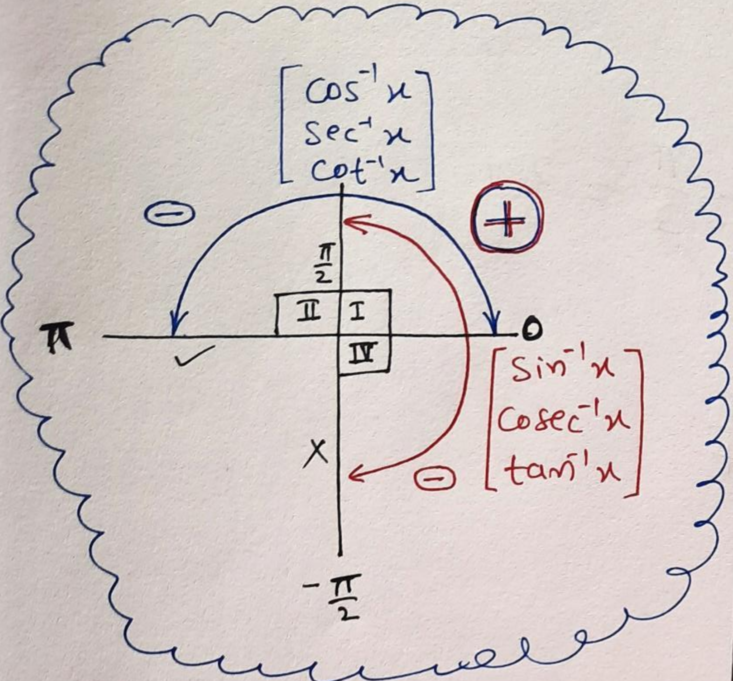
e.g. $\sec^{-1}(-\sqrt{2}) \to$ II-quadrant

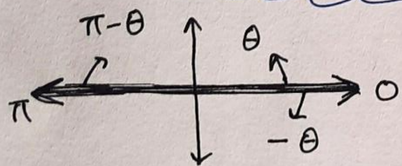
$\frac{\pi}{4}$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ ✓

Class 12 maths {Inverse Trigonometric Functions}

Exercise 2.1





Range ← $y = \sin^{-1} x$

ITF	Range
$\sin^{-1} x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[0, \pi]$
$\tan^{-1}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1}$	$(0, \pi)$

Exercise 2.1 (ITF)

Q.1. $\underbrace{\sin^{-1}\left(-\frac{1}{2}\right)}_{\text{IV quadrant}} = -\frac{\pi}{6}$ = output (angle)

Q.2 $\underbrace{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)}_{\text{I-quadrant}} = \frac{\pi}{6}$

Q.3 $\underbrace{\operatorname{cosec}^{-1}(2)}_{\text{I-quad.}} = \frac{\pi}{6}$

$\boxed{Q.4}$ $\underbrace{\tan^{-1}(-\sqrt{3})}_{\text{IV-quad.}} = -\frac{\pi}{3}$

$\boxed{Q.5.}$ $\underbrace{\cos^{-1}\left(-\frac{1}{2}\right)}_{\text{II-quad}} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\boxed{Q.6}$ $\underbrace{\tan^{-1}(-1)}_{\text{IV quad.}} = -\frac{\pi}{4}$

$\boxed{Q.7}$ $\underbrace{\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)}_{\text{I-quad.}} = \frac{\pi}{6}$

$\boxed{Q.8}$ $\underbrace{\cot^{-1}(\sqrt{3})}_{\text{I-quad.}} = \frac{\pi}{6}$

$\boxed{Q.9}$ $\underbrace{\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)}_{\text{II-quad.}} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\boxed{Q.10}$ $\underbrace{\operatorname{cosec}^{-1}(-\sqrt{2})}_{\text{IV quad.}} = -\frac{\pi}{4}$

$\boxed{Q.11}$ $\underbrace{\tan^{-1}(1)} + \underbrace{\cos^{-1}\left(-\frac{1}{2}\right)} + \underbrace{\sin^{-1}\left(-\frac{1}{2}\right)}$

$= \frac{\pi}{4} + \left(\pi - \frac{\pi}{3}\right) + \left(-\frac{\pi}{6}\right)$

$= \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$

$\boxed{Q.12}$ $\cos^{-1}\left(\frac{1}{2}\right) + 2 \times \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3} + 2 \times \frac{\pi}{6} = \frac{2\pi}{3}$

$\boxed{Q.13}$ If $\sin^{-1}x = y$, then –

$y = \sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ✓

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\boxed{Q.14}$ $\underbrace{\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)}$ is equal to –

(A) π (B) $-\frac{\pi}{3}$ ✓ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Ans. $\tan^{-1}(\sqrt{3}) - \left[\sec^{-1}(-2)\right]$

$= \frac{\pi}{3} - \left[\pi - \frac{\pi}{3}\right] = \frac{\pi}{3} - \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3} - \pi = -\frac{\pi}{3}$

Properties of Inverse Trigonometric Functions

Formula Sheet — Formula Hint

① (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$
(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$
(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$

~~tan~~ $\text{trigo}^{-1}\left(\frac{1}{x}\right) \longrightarrow \sin\theta = \frac{1}{\operatorname{cosec}\theta}$

② (i) $\sin^{-1}(-x) = -\sin^{-1}x$
(ii) $\tan^{-1}(-x) = -\tan^{-1}x$
(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

$\longrightarrow \sin(-\theta) = -\sin\theta$

$\text{trigo}^{-1}(-x)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$
(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$
(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

$\longrightarrow \cos(\pi-\theta) = -\cos\theta$

③ (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

$\longrightarrow \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

(Complementary Angle)

④ (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$
(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\longrightarrow \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A.\tan B}$

⑤ (i) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ — $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ — $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

* (iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ — $\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$

$\boxed{\text{all are in their respective Domains}}$

Proof:

① (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$; $x \in (-\infty, -1] \cup [1, \infty)$

let $\operatorname{cosec}^{-1} x = \theta = RHS$

$\Rightarrow x = \operatorname{cosec}\theta$

$\Rightarrow \frac{1}{x} = \sin\theta$

$\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \theta = LHS$

② (i) $\sin^{-1}(-x) = -\sin^{-1} x$

$LHS = \sin^{-1}(-x) = \theta$ (let)

$\Rightarrow -x = \sin\theta$

$\Rightarrow x = -\sin\theta$

$\Rightarrow x = \sin(-\theta)$

$\Rightarrow \sin^{-1}(x) = -\theta$

$\Rightarrow -\sin^{-1} x = \theta$

$RHS = \theta$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

$LHS = \theta = \cos^{-1}(-x)$

$\Rightarrow \cos\theta = -x$

$\Rightarrow -\cos\theta = x$

$\Rightarrow \cos(\pi - \theta) = x$

$\Rightarrow \pi - \theta = \cos^{-1} x$

$\Rightarrow \pi - \cos^{-1} x = \theta$

$RHS = \theta$

③ (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

let $\sin^{-1} x = \theta$

$\Rightarrow x = \sin\theta$

$\Rightarrow x = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$

$\Rightarrow \theta + \cos^{-1} x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(4) (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

let $\angle A$ $\quad \angle B$

$\tan^{-1}x = A$, $\tan^{-1}y = B$

$\Rightarrow x = \tan A$, $y = \tan B$

we know that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$\Rightarrow \tan(\tan^{-1}x + \tan^{-1}y) = \frac{x+y}{1-xy}$

$\Rightarrow \boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$ ☆

(5) (i) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

let $\tan^{-1}x = \theta \Rightarrow x = \tan\theta$

we know $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$\Rightarrow \sin(2\tan^{-1}x) = \frac{2x}{1+x^2}$

$\Rightarrow \boxed{2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)}$

e.g. Show that $\underbrace{\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{2}{11}\right)}_{} = \underbrace{\tan^{-1}\left(\frac{3}{4}\right)}_{}$

$$LHS = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{2}{11}\right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) \qquad \left\{\tan^{-1}(-x) = -\tan^{-1}x\right\}$$

$$\boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{11+4}{\cancel{2\times 11}}}{\frac{22-2}{\cancel{2\times 11}}}\right) = \tan^{-1}\left(\frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{20}}}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) = RHS$$

Exercise 2.2 (Chapter 2 - ITF)

Q.1 Prove that

$$\boxed{3\sin^{-1}x = \sin^{-1}(3x - 4x^3)}$$

$x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

we know that

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Let $\theta = \sin^{-1}x$ — (1)

$\Rightarrow \sin\theta = x$ — (2)

Substitute.

$\Rightarrow \sin(3\sin^{-1}x) = 3x - 4x^3$

$\Rightarrow \boxed{3\sin^{-1}x = \sin^{-1}(3x - 4x^3)}$

H.P.

Q.2 Prove that

$$\boxed{3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)}$$

$x \in \left[\frac{1}{2}, 1\right]$

we know that

$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

Let $\cos^{-1}x = \theta$

$\Rightarrow x = \cos\theta$

Substitute.

$\Rightarrow \cos(3\cos^{-1}x) = 4x^3 - 3x$

$\Rightarrow 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

H.P.

Q.3 Prove that $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

$$\boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

$x = 2/11$, $y = 7/24$

$\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11}\cdot\frac{7}{24}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Q.4 Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$\boxed{\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

$$\boxed{2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)}$$

$$\text{LHS} = \underbrace{2\tan^{-1}\left(\frac{1}{2}\right)}_{\text{Formula}} + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{2}\times\frac{1}{\cancel{2}}}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{\left(3/4\right)}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\underbrace{\frac{4}{3}}_{x}\right) + \tan^{-1}\left(\underbrace{\frac{1}{7}}_{y}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{\frac{1}{1}-\frac{4}{3}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{28+3}{\cancel{21}}}{\frac{21-4}{\cancel{21}}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right) = \text{RHS}$$

Exercise 2.2 Chapter - 2 (ITF)

Write the following functions in the simplest form:

$\boxed{Q.5}$ $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x \neq 0$

Substitution: $x = \tan\theta$ $\quad \tan^{-1}x = \theta$

$$= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin^2\frac{\theta}{2}}{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cdot\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$$

$\boxed{Q.6}$ $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $|x| > 1$

Substitution $x = \sec\theta \rightarrow \sec^{-1}x = \theta$

Pattern

$x^2 - 1$

$\downarrow$

$\sec^2\theta - 1 = \tan^2\theta$

$$= \tan^{-1}\left(\frac{1}{\sqrt{\sec^2\theta - 1}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\tan\theta) = \cancel{\cot^{-1}}\left(\cancel{\cot}\left(\frac{\pi}{2}-\theta\right)\right)$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1}x$$

$\boxed{Q.7}$ $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0<x<\pi$

$= \tan^{-1}\left(\sqrt{\frac{\cancel{2}\sin^2 x/2}{\cancel{2}\cos^2 x/2}}\right)$

$\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2\theta = 1-2\sin^2\theta$

$\cos x = 2\cos^2\frac{x}{2} - 1$

$1+\cos x = 2\cos^2 x/2$

$\cos x = 1-2\sin^2\frac{x}{2}$

$2\sin^2\frac{x}{2} = 1-\cos x$

$= \tan^{-1}\sqrt{\tan^2 x/2}$

$= \tan^{-1}\left(\tan\frac{x}{2}\right)$

$= \frac{x}{2}$ ✓

$\boxed{Q.8}$ $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $-\frac{\pi}{4} < x < \frac{3\pi}{4}$

Nr. & Dr. are divided by $\cos x$

$= \tan^{-1}\left(\frac{\left(\frac{\cos x - \sin x}{\cos x}\right)}{\left(\frac{\cos x + \sin x}{\cos x}\right)}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x . 1}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}.\tan x}\right) = \cancel{\tan^{-1}}\left(\cancel{\tan}\left(\frac{\pi}{4}-x\right)\right)$

$= \frac{\pi}{4} - x$

$\boxed{Q.9}$ $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right)$, $|x|<a$

$\dfrac{x}{a}$ बनाना है

~~Substitution~~ Nr. & Dr. both are divided by 'a'

$$= \tan^{-1}\left(\dfrac{\frac{x}{a}}{\frac{\sqrt{a^2-x^2}}{a}}\right) = \tan^{-1}\left(\dfrac{\frac{x}{a}}{\sqrt{\frac{a^2-x^2}{a^2}}}\right)$$

$$= \tan^{-1}\left(\dfrac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right)$$

$1-(\sin\theta)^2 = \cos^2\theta$

Substitution $\boxed{\dfrac{x}{a} = \sin\theta \text{ Let}} \Rightarrow \sin^{-1}\left(\dfrac{x}{a}\right) = \theta$

$$= \tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\dfrac{x}{a}\right)$$

Q.10 $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a>0$, $-\frac{a}{\sqrt3} < x < \frac{a}{\sqrt3}$

$x, a \rightsquigarrow \left(\frac{x}{a}\right)$ बनाना है

$x^3 \to \frac{x^3}{a^3}$

Nr. & Dr. are divided by (a^3)

$$= \tan^{-1}\left(\frac{\frac{3a^2x}{a^3} - \frac{x^3}{a^3}}{\frac{a^3}{a^3} - \frac{3ax^2}{a^3}}\right) = \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$$

Substitution $\boxed{\frac{x}{a} = \tan\theta}$ $\tan^{-1}\left(\frac{x}{a}\right) = \theta$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

Exercise 2.2 Chapter 2 ITF

Find the value of the each of the following

(11) $\tan^{-1}\left[2\cos\left(2\underbrace{\sin^{-1}\frac{1}{2}}\right)\right]$

$\sin^{-1}\frac{1}{2} = \theta \swarrow \frac{\pi}{6}$

$\frac{1}{2} = \sin\theta \swarrow 30^\circ = \frac{\pi}{6}$

$= \tan^{-1}\left(2\underbrace{\cos\left(2\times\frac{\pi}{6}\right)}\right)$

$= \tan^{-1}\left(\cancel{2}\times\frac{1}{\cancel{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ ✓

$\boxed{Q.12}$ $\cot(\underbrace{\tan^{-1}a + \cot^{-1}a}_{\frac{\pi}{2}}) = \cot\left(\frac{\pi}{2}\right) = \cot(90^\circ) = 0$

$\boxed{Q.13}$ $\tan\frac{1}{2}\left[\underbrace{\sin^{-1}\frac{2x}{1+x^2}}_{2\tan^{-1}x} + \underbrace{\cos^{-1}\frac{1-y^2}{1+y^2}}_{2\tan^{-1}y}\right]$ (By Prop.)

$= \tan\frac{1}{2}\left[2\tan^{-1}x + 2\tan^{-1}y\right]$

$= \tan\left(\tan^{-1}x + \tan^{-1}y\right)$ $\quad f(f^{-1}(x)) = x$

(by $\tan(A+B)$ formula)

$= \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\cdot\tan(\tan^{-1}y)} = \frac{x+y}{1-x\cdot y}$ ✓

Q.14 If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.

Ans.
$$\sin\left(\underbrace{\sin^{-1}\frac{1}{5} + \cos^{-1}x}_{\frac{\pi}{2}}\right) = 1$$

$$\sin(90^\circ) = 1$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

$$\therefore \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}(x) = \frac{\pi}{2}$$

Same

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$$

Property.

$$\boxed{x = \frac{1}{5}}$$

Q.15 If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, then find the value of x.

Ans.
$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

tan(LHS) = tan(RHS)

$$\Rightarrow \tan\left[\underbrace{\tan^{-1}\left(\frac{x-1}{x-2}\right)}_{A} + \underbrace{\tan^{-1}\left(\frac{x+1}{x+2}\right)}_{B}\right] = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{\frac{1}{1} - \left(\frac{x-1}{x-2}\right)\cdot\left(\frac{x+1}{x+2}\right)} = 1$$

$$\Rightarrow \frac{\dfrac{(x^2+x-2)+(x^2-x-2)}{\cancel{(x-2)(x+2)}}}{\dfrac{(x^2-4)-(x^2-1)}{\cancel{(x-2)(x+2)}}} = 1$$

$$\Rightarrow \frac{x^2+\cancel{x}-2+x^2-\cancel{x}-2}{\cancel{x^2}-4-\cancel{x^2}+1} = 1$$

$$\Rightarrow \frac{2x^2-4}{-3} = 1 \Rightarrow 2x^2-4 = -3$$

$$\Rightarrow 2x^2 = 4-3 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow \boxed{x = \pm\frac{1}{\sqrt{2}}}$$

Exercise 2.2

Q.16 Find the value $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

I-method:

$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ Solve

$\frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{3}$ ✓

$\sin\frac{2\pi}{3} = \sin 120^\circ = \frac{\sqrt{3}}{2}$ (+)

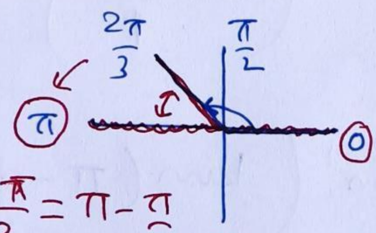
$\sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3}$

II-method,

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$

$\left(\because \sin^{-1}(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$= \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$



$\frac{2\pi}{3} = 120^\circ$

$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$

$\frac{2\pi}{3} = \pi - \frac{\pi}{3}$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$

$120^\circ = 180^\circ - 60$

$(\sin(\pi - \theta) = \sin\theta)$

$= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ✓✓

Q.17 $\tan^{-1}\left(\tan 3\frac{\pi}{4}\right)$ $\left(\frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}\right)$

I - method. $\tan^{-1}\left(\underbrace{\tan\frac{3\pi}{4}}_{\text{Solve.}}\right)$

$\tan\frac{3\pi}{4} = \tan 135^\circ = -1$

$\tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4}$

$= \underbrace{\tan^{-1}(-1)}_{\text{IV - quad.}}$

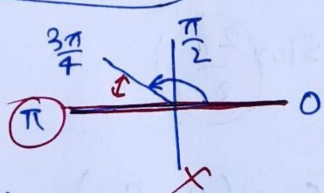
$=$

$= -\frac{\pi}{4}$ ✓

II - method. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq$ $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\boxed{\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$,

$= \tan^{-1}\left(\tan\frac{3\pi}{4}\right)$



$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$\tan(\pi - \theta) = -\tan\theta$

$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ✓

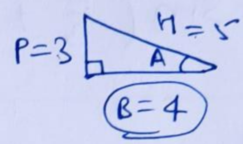
Q.18 $\tan\left(\underbrace{\sin^{-1}\frac{3}{5}}_{\angle A} + \underbrace{\cot^{-1}\frac{3}{2}}_{\angle B}\right) = \tan(A+B)$

$\sin^{-1}x$ = angle
$\tan^{-1}x$ = angle
$\text{trigo}^{-1}x$ = angle

Let $\sin^{-1}\frac{3}{5} = A$, $\cot^{-1}\frac{3}{2} = B \Rightarrow \frac{3}{2} = \cot B$

$\sin^{-1}\frac{3}{5} = A$

$\Rightarrow \frac{P}{H} = \frac{3}{5} = \sin A$ 10th



$\tan A = \frac{P}{B} = \frac{3}{4}$

$= \tan(A+B)$

$= \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$= \dfrac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}} = \dfrac{\frac{9+8}{\cancel{4\times 3}}}{\frac{12-6}{\cancel{4\times 3}}} = \frac{17}{6}$

Q.19 $\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$ is equal to (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

$\cos^{-1}\cos\left(\frac{7\pi}{6}\right) = \frac{7\pi}{6} \notin [0, \pi]$

$\boxed{\cos^{-1}x \in [0, \pi]}$ Range

$= \cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$= \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{6}\right)\right)$

π 0 $\cos(\pi+\theta) = -\cos\theta$

$\Rightarrow \cos^{-1}\left(-\cos\frac{\pi}{6}\right)$

$= \pi - \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$= \pi - \frac{\pi}{6}$

$= 5\pi/6$

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$\boxed{Q.20}$ $\sin\left[\frac{\pi}{3} - \underline{\sin^{-1}\left(-\frac{1}{2}\right)}\right]$ (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

$\underline{\sin(A-B)}$

$\underbrace{\sin^{-1}\left(-\frac{1}{2}\right)}_{\text{IV-quad.}} = -\frac{\pi}{6}$

$= \sin\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$\boxed{Q.21}$ $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ (A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

$\underbrace{\tan^{-1}(\sqrt{3})}$: angle $= \frac{\pi}{3}$ (I-quad.)

$\underbrace{\cot^{-1}(-\sqrt{3})}$: angle $= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ (II-quadrant)

$= \frac{\pi}{3} - \frac{5\pi}{6}$

$= \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$

Miscellaneous Exercise on Chapter (2)

Find the value of the following :→

(1) $\cos^{-1}\cos\frac{13\pi}{6}$ (2) $\tan^{-1}\tan\frac{7\pi}{6}$

---

(1) $\cos^{-1}\cos\underbrace{\frac{13\pi}{6}}_{x} \neq \frac{13\pi}{6}$ $\cos^{-1}x \in [0, \pi]$

$\frac{n\pi}{2}, \frac{n\pi}{3}, \frac{n\pi}{4}, \frac{n\pi}{6}$

(i) method.

$\underbrace{\cos^{-1}\cos\frac{13\pi}{6}}_{\text{Solve.}}$

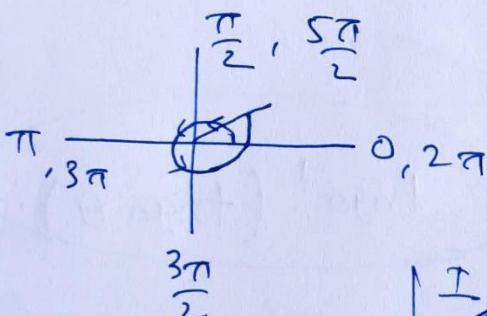
$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$

$= \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$ $\cos(2\pi + \theta) = \cos\theta$

$= \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

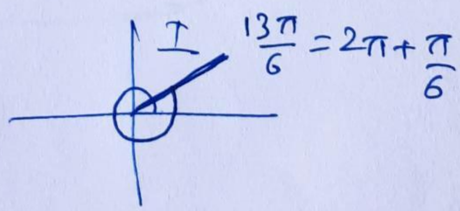
$= \frac{\pi}{6} \in [0, \pi]$

(ii) method.



$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

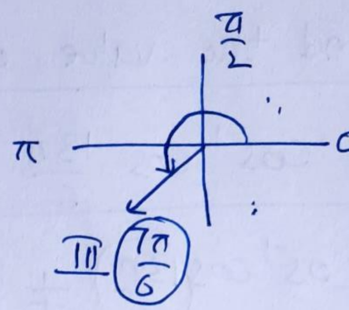
$= \cos^{-1}\left(\sqrt{3}/2\right)$



$= \frac{\pi}{6}$

Q.2 $\tan^{-1}\left(\tan \frac{7\pi}{6}\right) = \frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\boxed{\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$



$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$

(i) method

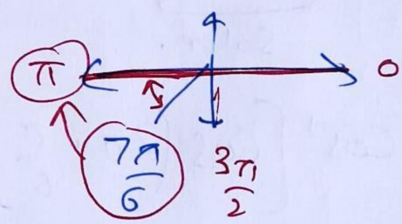
$\tan^{-1}\left(\tan \frac{7\pi}{6}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ ✓

$\left(\tan(\pi+\theta) = \tan\theta\right)$

II method

$\tan^{-1}\left(\tan \frac{7\pi}{6}\right)$

$= \tan^{-1}\left(\tan\left(\pi + \frac{\pi}{6}\right)\right)$



$= \cancel{\tan^{-1}}\left(\cancel{\tan}\left(\frac{\pi}{6}\right)\right)$

$\cancel{\frac{3\pi}{2} - \frac{\pi}{3}}$

$= \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan(\pi+\theta) = \tan\theta$

✓

$\text{trigo}^{-1}(\text{trigo}\,\theta) \rightarrow$ Range

Miscellaneous Exercise on Chapter 2

$\boxed{Q3 \text{ to } Q8}$

Q.3 $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

rough

$\tan\left(2\sin^{-1}\frac{3}{5}\right) = \frac{24}{7}$

$\to \theta$

$\sin^{-1}x = \theta$

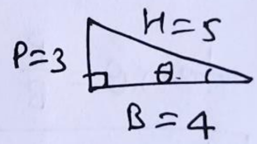
$\cos^{-1}x = \theta$ $\to$ Output angle

$\tan^{-1}x = \theta$

Let $\sin^{-1}\frac{3}{5} = \theta$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$\sin^{-1}\frac{3}{5} = \theta \Rightarrow \frac{3}{5} = \sin\theta = \frac{P}{H}$



$\tan\theta = \frac{P}{B} = \frac{3}{4}$

$$\Rightarrow \tan 2\theta = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$

$$\Rightarrow \tan 2\theta = \frac{3/2}{1-\frac{9}{16}} = \frac{3/\cancel{2}}{7/\cancel{16}_{\,8}} = \frac{24}{7}$$

$$\Rightarrow \tan 2\theta = \frac{24}{7} \Rightarrow 2\theta = \tan^{-1}\left(\frac{24}{7}\right)$$

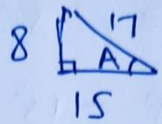
$$\Rightarrow \boxed{2\sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{24}{7}\right)}$$

Q.4 $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Let $\sin^{-1}\frac{8}{17}$ and $\sin^{-1}\frac{3}{5}$ be A, B (angles)

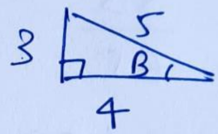
$\tan(A+B) = \frac{77}{36}$

$\sin^{-1}\frac{8}{17} = A \Rightarrow \frac{8}{17} = \sin A = \frac{P}{H}$



$\tan A = \frac{8}{15} = \frac{P}{B}$

$\sin^{-1}\frac{3}{5} = B \Rightarrow \frac{3}{5} = \sin B = \frac{P}{H}$



$\tan B = \frac{P}{Base} = \frac{3}{4}$

Now $\tan(A+B)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\tan(A+B) = \frac{\frac{8}{15} + \frac{3}{4}}{\frac{1}{1} - \frac{8}{15} \times \frac{3}{4}}$$

$$\Rightarrow \tan(A+B) = \frac{\frac{32+45}{\cancel{15\times 4}}}{\frac{60-24}{\cancel{15\times 4}}} = \frac{77}{36}$$

$$\Rightarrow A+B = \tan^{-1}\left(\frac{77}{36}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{77}{36}\right)$$

$\boxed{Q.5}$ $\underbrace{\cos^{-1}\frac{4}{5}}_{A} + \underbrace{\cos^{-1}\frac{12}{13}}_{B} = \cos^{-1}\frac{33}{65}$

Let A, B

$\cos(A+B) = \frac{33}{65}$

$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$\Rightarrow \cos(A+B) = \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$

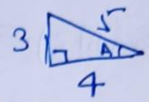
$\Rightarrow \cos(A+B) = \frac{48 - 15}{5 \times 13}$

$\Rightarrow \cos(A+B) = \frac{33}{65}$

$\Rightarrow A + B = \cos^{-1}\left(\frac{33}{65}\right)$

$\Rightarrow \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

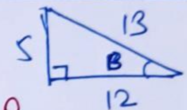
$\cos^{-1}\frac{4}{5} = A$



$\Rightarrow \frac{4}{5} = \cos A = \frac{B}{H}$

$\sin A = \frac{P}{H} = \frac{3}{5}$

$\cos^{-1}\frac{12}{13} = B$



$\Rightarrow \frac{12}{13} = \cos B = \frac{B}{H}$

$\sin B = \frac{P}{H} = \frac{5}{13}$

$\boxed{Q.6}$ $\underbrace{\cos^{-1}\frac{12}{13}}_{A} + \underbrace{\sin^{-1}\frac{3}{5}}_{B} = \sin^{-1}\left(\frac{56}{65}\right)$

$\sin(A+B) = \frac{56}{65}$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$\Rightarrow \sin(A+B) = \left(\frac{5}{13}\right) \times \left(\frac{4}{5}\right) + \left(\frac{12}{13}\right) \cdot \left(\frac{3}{5}\right)$

$\Rightarrow \sin(A+B) = \frac{20}{65} + \frac{36}{65}$

$\Rightarrow \sin(A+B) = \frac{56}{65}$

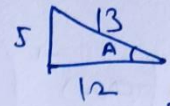
$\Rightarrow A + B = \sin^{-1}\left(\frac{56}{65}\right)$

$\Rightarrow \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Q.7 $\tan^{-1}\left(\frac{63}{16}\right) = \underbrace{\sin^{-1}\left(\frac{5}{13}\right)}_{A} + \underbrace{\cos^{-1}\left(\frac{3}{5}\right)}_{B}$

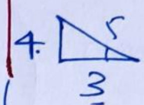
$$\frac{63}{16} = \tan(A+B)$$

$\sin A = \frac{5}{13} = \frac{P}{H}$



$\tan A = \frac{5}{12}$

$\cos B = \frac{3}{5} = \frac{B}{H}$



$\tan B = \frac{4}{3}$

Now $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\Rightarrow \tan(A+B) = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}}$$

$$\Rightarrow \tan(A+B) = \frac{\frac{15+48}{\cancel{12 \times 3}}}{\frac{36-20}{\cancel{12 \times 3}}} = \frac{63}{16}$$

$$\Rightarrow A+B = \tan^{-1}\left(\frac{63}{16}\right)$$

$$\Rightarrow \boxed{\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{63}{16}\right)}$$

$\boxed{Q.8}$ $\underbrace{\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)}+\underbrace{\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{8}\right)}=\frac{\pi}{4}$

$$\boxed{\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

LHS =

$$=\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{12}{\cancel{5\times7}}}{\frac{34}{\cancel{5\times7}}}\right)+\tan^{-1}\left(\frac{\frac{11}{\cancel{3\times8}}}{\frac{23}{\cancel{3\times8}}}\right)$$

$$=\tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\underbrace{\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)}_{\text{Formula}}$$

$$=\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{\cancel{\frac{325}{17\times23}}}{\cancel{\frac{325}{17\times23}}}\right)$$

$$=\tan^{-1}(1)=\frac{\pi}{4}=\text{RHS}$$

Miscellaneous Exercise on Chapter ②

Q 9, Q10
Q11, Q12

Prove that:

Q.9 $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0,1]$

Put $\boxed{x = \tan^2\theta}$

$\Rightarrow \tan^{-1}\sqrt{\tan^2\theta} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \tan^{-1}(\tan\theta) = \frac{1}{2}\cdot\cos^{-1}(\cos 2\theta)$

$\Rightarrow \theta = \frac{1}{2}(2\theta)$

$\Rightarrow \boxed{\theta = \theta}$

LHS = RHS.

Q.10 $\cot^{-1}\left(\frac{\sqrt{(1+\sin x)} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

$1+\sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$= \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2$

$1-\sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$= \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2$

$$LHS = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) \qquad x \in \left(0, \frac{\pi}{4}\right)$$

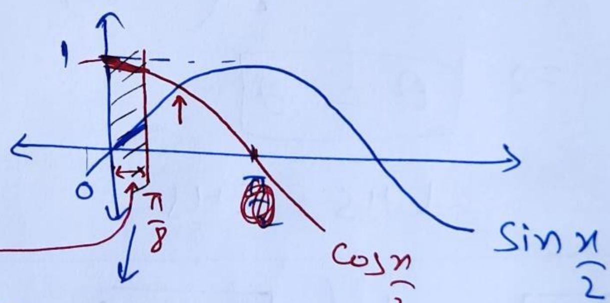
$$= \cot^{-1}\left\{\frac{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}+\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}-\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}\right\}$$

$\sqrt{x^2} = |x|$

$$= \cot^{-1}\left\{\frac{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|-\left|\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)\right|}\right\}$$

$x \in \left(0, \frac{\pi}{4}\right)$

$\frac{x}{2} \in \left(0, \frac{\pi}{8}\right)$



$\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|$

$\cos\frac{x}{2} > \sin\frac{x}{2}$

$|(\text{Small} - \text{Big})|$

$|\text{negative}| \implies -(\text{negative}) \to -\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)$

$$= \cot^{-1}\left\{\frac{\cancel{\sin\frac{x}{2}}+\cos\frac{x}{2}-\cancel{\sin\frac{x}{2}}+\cos\frac{x}{2}}{\sin\frac{x}{2}+\cancel{\cos\frac{x}{2}}+\sin\frac{x}{2}-\cancel{\cos\frac{x}{2}}}\right\}$$

$$= \cot^{-1}\left\{\frac{\cancel{2}\cos\frac{x}{2}}{\cancel{2}\sin\frac{x}{2}}\right\} = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2} = \text{RHS}.$$

$\boxed{Q.11}$ $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x$

$\underline{-\frac{1}{\sqrt{2}} \le x \le 1}$ $\underline{[\text{Hint: } x = \cos 2\theta]}$

Solution : LHS $= \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$x = \cos 2\theta$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$\cos 2\theta \to 2\cos^2\theta - 1$, $\cos 2\theta \to 1 - 2\sin^2\theta$

$$= \tan^{-1}\left(\frac{\sqrt{\cancel{2}\cos^2\theta}-\sqrt{\cancel{2}\sin^2\theta}}{\sqrt{\cancel{2}\cos^2\theta}+\sqrt{\cancel{2}\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{(\cos\theta - \sin\theta)/\cos\theta}{(\cos\theta + \sin\theta)/\cos\theta}\right)$$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$\dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B} = \tan(A-B)$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\cdot\tan\theta}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right)$

$= \frac{\pi}{4} - \theta$

$x = \cos 2\theta$

$\cos^{-1} x = 2\theta$

$\frac{1}{2}\cos^{-1} x = \theta$

$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$

$=$ RHS.

$\boxed{Q.12}$ $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ Prove

LHS $= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

Property

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right]$

$= \frac{9}{4}\left[\underbrace{\cos^{-1}\left(\frac{1}{3}\right)}_{\theta}\right]$

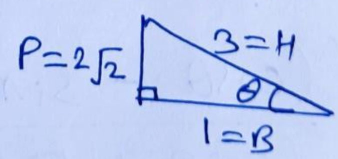
$= \frac{9}{4}\theta$

$= \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$=$ RHS

Let $\cos^{-1}\frac{1}{3} = \theta$

$\frac{B}{H} = \frac{1}{3} = \cos\theta$



$P^2 + 1 = 9$

$P^2 = 8$

$P = 2\sqrt{2}$

$\sin\theta = \frac{2\sqrt{2}}{3}$

$\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Miscellaneous Exercise on Chapter (2)

$\boxed{Q\ 13 \text{ to } Q\ 17}$

Solve the following equations:

$\boxed{Q.13}$ $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$ $\quad x = ?$

ITF → Angle
$\sin^{-1}x \to$ angle
$\cos^{-1}x \to \theta$
$\tan^{-1}x \to \theta$

$\Rightarrow \tan\left[2\tan^{-1}(\cos x)\right] = 2\operatorname{cosec} x$

$\tan[2\theta] = \dfrac{2\tan(\theta)}{1-\tan^2\theta}$

Formula

$\Rightarrow \dfrac{\cancel{2}\tan(\tan^{-1}\cos x)}{1-\left[\tan(\tan^{-1}\cos x)\right]^2} = \cancel{2}\operatorname{cosec} x$

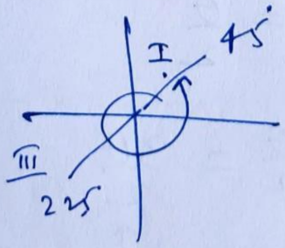
$\Rightarrow \dfrac{\cos x}{1-\cos^2 x} = \dfrac{1}{\sin x}$

$\Rightarrow \dfrac{\cos x}{\sin^{\cancel{2}} x} = \dfrac{1}{\cancel{\sin x}}$

$n \in I$

$\boxed{\begin{array}{l}\tan x = \tan\theta \\ x = n\pi + \theta\end{array}}$

$\Rightarrow \dfrac{\cos x}{\sin x} = 1$



$\Rightarrow \dfrac{\sin x}{\cos x} = 1$

$\tan x = 1 = \tan\dfrac{\pi}{4}$

$\Rightarrow \boxed{\tan x = 1}$ General Solution $\boxed{x = n\pi + \dfrac{\pi}{4}}$ $n \in \mathbb{Z}$

$\boxed{Q.14}$ $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, $(x>0)$

$x = ?$

I-method

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}(x)$$

II-method $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

(let $x = \tan\theta$)

$$\Rightarrow \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta\cdot 1}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right) = \frac{1}{2}(\theta)$$

$\tan(A-B)$

$$\Rightarrow \cancel{\tan^{-1}}\left(\cancel{\tan}\left(\frac{\pi}{4}-\theta\right)\right) = \frac{\theta}{2}$$

$$\Rightarrow \frac{\pi}{4} - \theta = \frac{\theta}{2}$$

$$\Rightarrow \frac{\pi}{4} = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{6}}$$

$\theta = \frac{\pi}{6}$

Solve $x = ?$

$x = \tan\theta$

$x = \tan\frac{\pi}{6}$

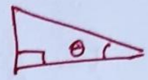
$x = \frac{1}{\sqrt{3}}$ ☆

Q.15 $\sin(\tan^{-1}x)$, $|x|<1$ is equal to –

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) ✓ $\frac{x}{\sqrt{1+x^2}}$

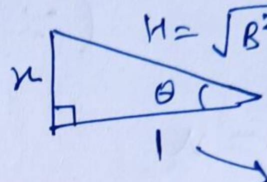
Ans. $\sin(\underbrace{\tan^{-1}x}_{\text{angle} = \theta \text{ (let)}})$

$\boxed{\text{trigo}_1(\underbrace{\text{trigo}_2^{-1}\, x}_{\theta})}$



$\tan^{-1}x = \theta$

$\Rightarrow \frac{x}{1} = \tan\theta = \frac{\text{Perp.}}{\text{Base}}$



$H = \sqrt{B^2+P^2} = \sqrt{1+x^2}$

$\sin\theta = \frac{P}{H} = \frac{x}{\sqrt{1+x^2}}$

Question $= \sin(\tan^{-1}x)$

$= \sin(\theta)$

$= \frac{x}{\sqrt{1+x^2}}$

Q.16 $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to –

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) ✓ 0 (D) $\frac{1}{2}$

Sol^n. $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$

$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$

$\Rightarrow \cos(-2\sin^{-1}x) = 1-x$

Prop.

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

$\sin^{-1}x \to \theta$

$\cos(-\theta) = \cos\theta$

$\Rightarrow \cos(2\underbrace{\sin^{-1}x}) = 1-x$

let $\sin^{-1}x = \theta$ (angle) $\Rightarrow x = \sin\theta$

$\Rightarrow \cos(2\theta) = 1-x$

$\cos 2\theta \to \cos^2\theta - \sin^2\theta$

$\cos 2\theta \to 2\cos^2\theta - 1$

$\cos 2\theta \to \boxed{1-2\sin^2\theta}$

$\cos 2\theta \to \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\Rightarrow 1 - 2\sin^2\theta = 1-x$

$\Rightarrow \cancel{1} - 2(x)^2 = \cancel{1} - x$

$(\because \sin^{-1}x = \theta \Rightarrow x = \sin\theta)$

$\Rightarrow 2x^2 = x$

$\Rightarrow 2x^2 - x = 0$

$\Rightarrow x.(2x-1) = 0$

$\boxed{x=0}$

$2x-1=0 \Rightarrow \boxed{x=\frac{1}{2}}$

original equation:

$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$\boxed{x=0}$ Check

$\Rightarrow \sin^{-1}(\underline{1-0}) - 2\sin^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} - 0 = \frac{\pi}{2}$

LHS = RHS

$x=0$

$\boxed{x=\frac{1}{2}}$ Check

$\sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right) \neq \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{6} - 2\left(\frac{\pi}{6}\right) \neq \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{6} \neq \frac{\pi}{2}$

Q.17 $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to –

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

✓ (C) $\frac{\pi}{4}$ (D) $-\frac{3\pi}{4}$

(i)- method : $\frac{x}{y} = \tan\theta$

(ii) $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

Property $\boxed{\tan^{-1}(X) - \tan^{-1}(Y) = \tan^{-1}\left(\frac{X-Y}{1+XY}\right)}$

$$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{\frac{1}{1} + \frac{x}{y}\cdot\frac{x-y}{x+y}}\right) = \tan^{-1}\left[\frac{\frac{x^2 + \cancel{xy} - \cancel{xy} + y^2}{\cancel{y(x+y)}}}{\frac{\cancel{xy} + y^2 + x^2 - \cancel{xy}}{\cancel{y\cdot(x+y)}}}\right]$$

$$= \tan^{-1}\left(\cancel{\frac{x^2+y^2}{y^2+x^2}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$